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Enhanced Fixed Point Theorems for Compatible and Semi Compatible Mappings of Type α and β In Menger Spaces

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ABSTRACT

This paper presents advanced fixed-point theorems for both compatible and semi-compatible mappings of types α and β within Menger spaces. We extend classical results by exploring the conditions under which these mappings exhibit enhanced fixed-point properties. Our findings include new theorems with detailed proofs, contributing to a deeper understanding of the interplay between compatibility conditions and fixed-point existence. These results have potential applications in various mathematical and applied fields.

Keywords- Fixed Point Theorems, Compatible Mappings, Semi-Compatible Mappings, Menger Spaces, Type α and β.

2020 hematics Subject Classification- Primary 47H10; Secondary 54H25, 54H20, 54E50, 54H99.

I. INTRODUCTION

Fixed point theorems are crucial in mathematical analysis and topology, with significant contributions from Karl Menger. Menger's introduction of Menger spaces, which generalizes metric spaces, has been foundational in extending fixed point theory to more complex structures [1]. His work has inspired a broad range of studies on fixed points in various types of spaces [2]. In recent years, significant progress has been made on fixed point theorems for compatible mappings in these spaces [3, 4, 5]. Despite this, there remains a notable gap in the exploration of fixed-point properties for mappings of types α and β, particularly when considering their interaction with semi-compatible mappings [6, 7, 8]. Recent studies have addressed certain aspects but have not fully integrated these types into a cohesive theory [9, 10, 11]. This paper addresses this gap by providing enhanced fixed-point theorems that encompass both compatible and semi-compatible mappings of types α and β within Menger spaces, offering new theoretical insights and potential applications [12, 13, 14]. These contributions are poised to advance the understanding of fixed-point phenomena in generalized metric spaces and open new avenues for further research.

II. PRELIMINARIES: BASIC CONCEPTS AND SOME DEFINITIONS

Menger space: A Menger space is a generalization of a metric space where the distance function is not necessarily a metric but satisfies certain axioms that generalize those of metric spaces. formally, let X be a set and $d: X \times X \to [0, \infty)$ be a function. then (X, d) is called a Menger space if d satisfies the following conditions for all $x, y, z \in X$:
PM-1: **Non-negativity;** $d(x, y) \geq 0$.
PM-2: Symmetry: $d(x, y) = d(y, x)$.
PM-3: Triangle Inequality: There exist a function $\boldsymbol{f}: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ such that

$$d(x, z) \leq f(d(x, y) + d(y, z)$$

Compatible Mappings: Let (X, d) ne a Menger space. Two mappings $T: X \times X \to X$ and $S: X \times X \to X$ are said to be compatible if pair of points $(x, y) \in X$,then the following condition holds: $d(T(x), T(y) \leq f(d(x, y))$. Where f is a function as defined in the triangle inequality of the Menger space .this condition ensures that T and S do not ''diverge'' too much from other.

Semi-Compatible Mappings; Two mappings $T: X \times X \to X$ and $S: X \times X \to X$ are called semi-compatible if they satisfy the condition:

$$d(T(x), S(x)) \leq f(d(x, S(x)))$$

For all$x \in X$, where f is as defined in the triangular inequality. This condition captures a weaker form of compatibility compared to the full compatibility.

Type α and β Mappings:

(1) **Type ''α'' Mapping:** A mapping $T: X \times X \to X$ is of type ''α'' if it satisfies the following condition: $d(T(x), T(y)) \leq \boldsymbol{\alpha}.\mathrm{d(x, y)}$,for all $x, y \in X$ where α is a constant with $0 < \alpha < 1$. This definition is inspired by contraction in metric space.

(2) **Type '' β'' Mapping:** A mapping $T: X \times X \to X$ is of type '' **β**'' if it satisfies the following condition: $d(T(x), T(y)) \leq \boldsymbol{\beta}.\min\{\mathrm{d}(x, y), d(x, T(y)), d(T(x), y)\}$, for all $x, y \in X$ where **β** is a constant with $0 \leq \boldsymbol{\beta} < 1$.This definition generalizes the concept of contraction to a broader class of function .

Fixed point: A point $x \in X$ is called a fixed f a mapping $T: X \times X \to X$ if $T(x) = x$. Fixed point theorems often aim to establish the existence of such point under certain conditions on the mapping involved.

Definition 2.1.[15] A left continuous and non-decreasing function $F: R \to R^+$,is said to be distribution function if it's $inf_{x \in R} F(x) = 0$ and $sup_{x \in R} F(x) = 1$.

Definition 2.2.[15] An ordered pair (X, F) is said to be Probabilistic Metric space shortly knowns as (PM-space), where X be an abstract set of elements and $F: X \times X \to X$.

PM 1. $F_{(x,y)} = 0$, for all $x, y \in X$.

PM 2. $F_{(x,y)} = 1$, for all $x > 0$, $if\ and\ only\ if\ x = y$.

PM 3. $F_{(x,y)} = F_{(y,x)}$, for all $x, y \in X$.

PM 4. For all $x, y, z \in X$ and for all $x, y > 0, F_{(x,y)} = 1, F_{(y,z)} = 1 \Longrightarrow F_{(x,z)}(x + y) = 1$.

Here, $\mathrm{F}_{(x,y)}$ represents the value of $\mathrm{F}_{(x,y)}$ at $x \in X$

Definition 2.3.[15] Menger space or Menger Probabilistic Metric Space, is a triplet (X, F, T),where (X, F) is a PM space and T is a triangular norm which satisfies the condition:

PM 5. $F_{(x,z)}(x + y) \geq T(F_{(x,y)}, F_{(y,z)}.)$ For all $x, y, z \in X$.

Definition 2.4.[15] A mapping $f: X \to X$ in Menger space (X, F, t), is said to be Continuous at a point $x \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there exist $\varepsilon_1 > 0$ and $\lambda_1 > 0$ such that if $\mathrm{F}_{(x,y)}(\varepsilon_1) > \lambda_1$, then $\mathrm{F}_{(fx,fy)}(\varepsilon) > 1 - \lambda$.

Definition 2.5.[15] Let X be a non-empty set and $S, T: X \to X$ be arbitrary mapping, then $x \in X$ is said to be a common fixed point of $S\ and\ T$ if $S(x) = T(x) = x$ for all $x \in X$.

Definition 2.6.[15] Two mapping said to be **Compatible Mapping** in Menger space (X, F, T) *iff* $\lim_{n\uparrow\infty} F_{STx_n, STx_n}(x)=1$ for all $x > 0$.

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some t in X.

Definition 2.7.[16] A function $F(t): (-\infty, +\infty) \to [0,1]$ is called a distance distribution function if it non-decreasing and left -continuous with limit $t \to \infty F(t) = 0, limit\ t \to \infty F(t) = 1$ and $F(0)$.The set of all distance distribution functions is denoted by D^+. A special Menger distance distribution function is given by

$$H(t) = \begin{cases} 0 & , t \leq 0, \\ 1, & t > 0. \end{cases}$$

Example 1 Fixed Point in a Metric Space

Consider the metric space (X, d) where $X = [0,1]$ and d is the usual absolute value metric

$$d(x, y) = |x - y|.$$

Let $T: X \to X$ be defined by $T(x) = \frac{x}{2}$.

If T is of type α:

Here, α = ½, for any $x, y \in X$,

$d(T(x), T(y)) = \left|\frac{x}{2} - \frac{y}{2}\right| = \frac{1}{2}|x - y| = \frac{1}{2} d(x, y)$.

Thus T is of type $\alpha = \frac{1}{2}$.

Now find the fixed point $T(x) = \frac{x}{2} = x \Longrightarrow 0$.

Thus $x = 0$ is the fixed point of T.

Example 2 Fixed Point in Generalized Metric Space
Consider the Menger space (X, d) where $X = \mathbb{R}^2$ and d is defined by $d((x_1, y_1), (x_2, y_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, a generalized form.
Let Let $T: X \to X$ be defined by $T(x, y) = (0.5x, 0.5y)$.
Now check for compatibility with Let $S(x, y) = (0.5x, 0.5y)$:
Here $d(T(x, y), S(x, y) = d\big((0.5x, 0.5y), (0.5x, 0.5y)\big) = 0$.
Clearly $d(T(x, y), S(x, y) \le f\left(d\big((x, y), (x, y)\big)\right) = 0$.
This show that S *and* T are compatible in this generalized metric.
Now again for fixed point $T(x, y) = (x, y)$:
$(0.5x, 0.5y) = (x, y) \Longrightarrow x = 0$ *and* $y = 0$.
Thus (0,0) is the fixed point of T.
Example 3 Semi-Compatible Mapping
Consider the Menger space (X, d) where $X = [0,1]$ and d is defined as in example
Let $T(x) = x + \frac{1}{2}$ *and* $S(x) = \frac{x}{2}$.
Now check if *and* T are semi-compatible:
We need to verify $d(T(x), S(x)) \le f\big(d(x, S(x))\big) = 0$.
Here $d\big(T(x), S(x)\big) = \left|\left(x + \frac{1}{2}\right) - \frac{x}{2}\right| = \left|\left(\frac{x}{2} + \frac{1}{2}\right)\right|$. For $x \in [0,1]$,
$d\big(x, S(x)\big) = d\left(x, \frac{x}{2}\right) = \left|\left(x - \frac{x}{2}\right)\right| = \frac{x}{2}$.
Let $f\big(d(x, S(x))\big) = \frac{x}{2} + \frac{1}{2} \le f\left(\frac{x}{2}\right)$.
Now obtain fixed point:
$T(x) = x$:
$x + \frac{1}{2} = x \Longrightarrow \frac{1}{2} = 0$.
Since there are no solution, T has no fixed point in this space.
Lemma 2.1. Let (X, d) be a Menger space, if $T: X \to X$ is a mapping of type 'α with $0 \le \alpha < 1$ and is compatible with itself, then T has at least one fixed point in X.
Proof; **Type $\boldsymbol{\alpha}$ Mapping:** we know that A mapping T is of type α if there exist $\alpha \in [0,1)$ such that for all $x, y \in X$, $d\big(T(x), T(y)\big) \le \boldsymbol{\alpha}. d(x, y)$.
Compatibility; T is compatible with itself if for $x, y \in X$,
$d\big(T(x), T(y)\big) \le \boldsymbol{f} d(x, y))$. Where f is a function related to the generalized triangle inequality in Menger spaces.
Now **constructing a sequence: Suppose** $\{\boldsymbol{x_n}\}$ **in** X by setting $x_0 \in X$ and $x_{n+1} = T(x_n)$ for $n \ge 0$.
Now **show that the sequence is Cauchy:** for $m > n$,
$d(x_{n+m}, x_n) = d\big(T^m(x_n), T^m(x_n)\big)$.
By applying the type $\boldsymbol{\alpha}$ property iterative,
$d(x_{n+m}, x_n) \le \boldsymbol{\alpha}^m d(x_n, x_n) = 0$.
Thus
$d(x_{n+m}, x_n) \le \boldsymbol{\alpha}^m d(x_n, x_{n+1}) \le \boldsymbol{\alpha}^m d(x_n, x_{n+1})$.
Since $\alpha < 1, \boldsymbol{\alpha}^m \to 0$ *as* $m \to \infty$ *make* $\{x_n\}$ a Cauchy sequence in X .
Now again Existence of a limit:
Since X is a Menger space (assuming completeness for the context), the Cauchy sequence $\{x_n\}$ converges to a point $x \in X$.
Now verify the fixed point by continuity of T.
⟹ The mapping *T* has at least one fixed point in *X*.
Lemma 2.2. Fixed Point for Semi-Compatible Mapping of Type β
Let (X, d) be a Menger space. If $T: X \to X$ is a mapping of type β with $0 \le \beta < 1$ and T is semi-compatible with itself, then T has at least one fixed point in X.
Proof: Type β Mapping: A mapping T is of type **β if there exist β** $\in [0,1)$ such that for
$x, y \in X, d\big(T(x), T(y)\big) \le \boldsymbol{\beta}. \min\{d(x, y), d\big(x, T(y)\big), d\big(T(x), y\big)\}$,
Semi-Compatibility: T is semi-compatible with itself if for all $x \in X$,
$d\big(T(x), T(x)\big) \le \boldsymbol{f} d(x, T(x)))$,
Where f is as defined in the generalized inequality of the Menger space.
Now constructing a sequence: Defined a sequence $\{x_n\}$ in X by putting $x_0 \in X$ and $x_{n+1} = T(x_n)$ for $n \ge 0$.
Now **show that the sequence is Cauchy:** for $m > n$,
$d(x_{n+m}, x_n) = d\big(T^m(x_n), T^m(x_n)\big)$.
Using the type,β *property* ,

$d(x_{n+m}, x_n) \leq \boldsymbol{\beta}. d(x_n, x_n) = 0,$
Iteratively $n = 1,2,3 \dots \dots n = m$
$d(x_{n+m}, x_n) \leq \boldsymbol{\beta}^2. d(x_n, x_n) = 0,$
$d(x_{n+m}, x_n) \leq \boldsymbol{\beta}^3. d(x_n, x_n) = 0,$
$d(x_{n+m}, x_n) \leq \boldsymbol{\beta}^m. d(x_n, x_n) = 0,$
Since $\beta < 1, \boldsymbol{\beta}^m \to 0. as\ m \to \infty$,making $\{x_n\}$ a Cauchy sequence in X.
Existence of a Limit: Since X is complete, then the Cauchy sequence $\{x_n\}$ converges to some point $x \in X$,
Now verify the Fixed point: By semi-compatibility property and the limit process,
$\boldsymbol{x = \lim_{n\to\infty} T(x) = T(\lim_{n\to\infty} x_n = T(x)}.$
Thus x is a fixed point in X.
Therefore, the mapping T has at least one fixed point in X.

III. MAIN RESULTS

In this section we prove Two theorems
Theorem 3.1. Let (X, F, T) be a compatible Menger space, and let $S, T: X \to X$ be two mappings of Type **α with 0** $\leq \alpha < 1$ .Assume the following conditions:
1. ***S* and *T*** are compatible mappings.
2. There exist $\alpha \in [0,1]$ such that for all $x, y \in X$, $d(S(x), T(y)) \leq \alpha d(x, y)$.
3. Both S and T are continuous mappings
4. There exists a common fixed point $z \in X$ such that $S(z) = T(z) = z$.

Then ***S* and *T* have unique common fixed point in *X*.**
Proof: We begin noting the (X, F, T) is a complete Menger space, which means that X is a probabilistic metric space where the metric satisfies the triangle inequality in a probabilistic sense, governed by the triangular norm T.
Let $x_0 \in X$ be an arbitrary point in the space. we will construct a sequence $\{x_n\}$ in X defined by the iterative process: $x_{n+1} = S(x_n)$ for all $n \geq 0$.
Our goal is to demonstrate that this sequence converges to a point $x \in X$,which will be a common fixed point of the mapping ***S* and *T* .**
Step 1: Establishing the Contraction Condition
Given that *S* and *T* are mappings of type α, there exist a constant $\alpha \in [0,1)$ such that for all $x, y \in X$,
$d(S(x), T(y)) \leq \alpha d(x, y).$ .... (1)
This inequality implies that ***S* and *T*** bring points together in a probabilistic sense, with the contraction factor α dictating the rate of convergence.
Step 2: Constructing the Sequence
Now, consider the sequence $\{x_n\}$ defined by $x_{n+1} = S(x_n)$ .... (2)
Applying the contraction condition, we have from equation (1)
$d(S(x_{n+1}), T(x_n)) \leq \alpha d(x_{n+1}, x_n).$ .... (3)
Since $x_{n+1} = S(x_n)$ and $x_n = S(x_{n-1})$, we can rewrite the (3) we get
$d(S(S(x_n), T(S(x_{n-1}))) \leq \alpha d(S(x_n), S(x_{n-1})).$ .... (4)
Step 3: Compatibility Condition and Convergence
The compatibility condition between S and T means that for any sequence $\{x_n\}$ in X.if $\lim_{n\to\infty} x_n = x$, then:
$\lim_{n\to\infty} F(ST(x_n), TS(x_n)) = 1,$ .... (5)
This implies that the distance between $ST(x_n)$ and $TS(x_n)$ in the probabilistic metric sense tends to zero as n increases.
Step 4: Cauchy Sequence and Completeness of *X*
We now show that the sequence $\{x_n\}$ is Cauchy. Consider two terms in the sequence, say $x_m$and x_n with $m > n$. By repeatedly applying the contraction condition, we get:
$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots \dots. +d(x_{n+1}, x_n)$ .... (6)
Applying the type α property iteratively, we have
$d(x_m, x_n) \leq \alpha^{m-n} d(x_n, x_{n-1}) + \alpha^{m-n-1} d(x_{n-1}, x_{n-2}) + \cdots \dots + d(x_{n+1}, x_n)$ ..... (7)
Since $\alpha < 1$, as $m, n \to \infty, \alpha^{m-n} \to 0$, making the series converge. Thus $\{x_n\}$ is a Cauchy sequence.
By the completeness of X ,there exist a limit $x \in X$ such that:
$\lim_{n\to\infty} x_n = x.$ .... (8)
Step 5: Existence of the Fixed Point
Now we verify that x is indeed a fixed point of both S and T.By the continuity of S and T ,and since $x_n \to x$, we have:
$\lim_{n\to\infty} S(x_n) = S(\lim_{n\to\infty} x_n) = S(x),$.. (9)

$\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(x)$,………………………………………………….. (10)
Since the sequence $\{x_n\}$ satisfy by (2) $x_{n+1} = S(x_n)$, we also have:
$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} S(x_n) = S(x)$. ……………………………………………… (11)
Similarly, $\lim_{n\to\infty} T(x_n) = T(x)$ …………………………………………………………………… (12)
Thus $S(x) = T(x) = x$, proving that x is a common fixed point of S and T.
Step 6: Uniqueness of the Fixed Point
Finally, we establish the uniqueness of the fixed point. Suppose y is another common fixed point such that $S(y) = T(y) = y$. Then, using the type α property:
$d(S(x), S(y)) \le \alpha d(x, y)$ …….. ………………………………………… (13)
Similarly,
$d(T(x), T(y)) \le \alpha d(x, y)$ ….……………………………………………………… (14)
$S(x) = T(x) = x$, and $S(y) = T(y) = y$ we have:
$d(x, y) \le \alpha d(x, y)$. ….…………………………………………………(15)
Given that $\mathbf{0} \le \alpha < 1$,the above inequality implies that $d(x, y) = 0$,hence $x = y$.
Thus, the fixed point x is unique.
Theorem 3.2. Theorem 3.1. Let (X, F, T) be a compatible Menger space, and let $S, T: X \to X$ be semi-compatible mappings of Type $, \beta$ **with** $\mathbf{0} \le, \beta < 1$ .Assume the following conditions:
1.***S* and *T*** are semi-compatible mappings.
2. There exist,$\beta \in [0,1]$ such that for all $x, y \in X$, $d(S(x), T(y)) \le, \beta d(x, y)$.
3.Both S and T are continuous at their common fixed point.
4. There exists a point $z \in X$ such that $S(z) = T(z) = z$.
Then ***S* and *T* have a common fixed point in *X*.**
Proof: Step 1: Definition and Initial Conditions
We start by defining the concepts involved. In a Menger space (X, F, T) ,the function F represents a probabilistic distance between points, S and T is a triangular norm ensuring the probabilistic triangular inequality satisfied.
The mapping ***S* and *T* are of type** $, \beta$ **if there exist** $, \beta \in [0,1)$ such that for all $x, y \in X$,
$d(S(x), T(y)) \le \beta d(x, y)$.
This condition implies that S and T do not increase the distance between points by more than a factor of β, which is less than 1, ensuring contraction in the probabilistic sense.
Step 2: Sequence Construction
Choose an arbitrary point $x_0 \in X$ and defined a sequence $\{x_n\}$ by
$x_{n+1} = S(x_n)$, for all $n \ge 0$.
Our aim is to show that this sequence $\{x_n\}$ converges to a common fixed point of S and T.
Step 3: Applying the Contraction Condition
$(S(x_{n+1}), T(x_n)) \le \beta d(x_{n+1}, x_n)$.
Since $x_{n+1} = T(x_n)$ and $x_n = T(x_{n-1})$,This becomes:
$d(S(T(x_n), T(T(x_{n-1}))) \le \beta d(T(x_n), T(x_{n-1}))$
Given that S and T are semi-compatible, the distance between the iterates under S and T. becomes increasingly small as the progresses.
Step 4: Cauchy sequence and completeness of *X*
Now we show that $\{x_n\}$is a Cauchy sequence. consider two terms x_m and $x_n$with the condition
$m > n$. applying the contraction condition iteratively
$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots \ldots . . + d(x_{n+1}, x_n)$
Using the type β property repeatedly, we get
$d(x_m, x_n) \le \beta^{m-n} d(x_n, x_{n-1}) + \beta^{m-n-1} d(x_{n-1}, x_{n-2}) + \cdots \ldots + d(x_{n+1}, x_n)$
Since $\beta < 1$, the term involving β^{m-n} tends to zero as $m, n \to \infty$, making the sequence $\{x_n\}$ is a Cauchy.
By the completeness of X, the sequence $\{x_n\}$ converges to some point $x \in X$ such that
$\lim_{n\to\infty} x_n = x$.
Step 5; Existence of the Fixed Point
We now verify x is indeed a fixed point of both S and T.By the continuity of S and T at the common fixed point z, since since $x_n \to x$, we have:
$\lim_{n\to\infty} S(x_n) = S(\lim_{n\to\infty} x_n) = S(x)$, $\lim_{n\to\infty} T(x_n) = T(\lim_{n\to\infty} x_n) = T(x)$,
Since the sequence $\{x_n\}$ is constructed by the recursive relation $x_{n+1} = T(x_n)$,we also have;
$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} T(x_n) = T(x)$.

Thus $T(x) = x$.
Step 6: Semi-Compatibility and Common Fixed point
Since S and T are semi-compatible, we know that:
$\lim_{n\to\infty} F(TS(x_n), TS(x_n) = 1$.
Given that $T(x) \to x$, we have $S(x) = T(x)$. Therefore:
$S(x) = T(x) = x$,
Which shows that x is a common fixed point of both S and T.

IV. CONCLUSION

This paper presents enhanced fixed-point theorems for compatible and semi-compatible mappings of types α and β in Menger spaces. Our results extend the existing theory by providing new insights into the behavior of these mappings and their fixed points. These theorems offer valuable contributions to both theoretical and applied fields. Future research can build on these findings to explore further applications and generalizations.

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